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A renormalisation group technique for spin and gauge systems with topological excitations

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Abstract. We present a real space renormalisation group technique able to deal explicitly with gases of topological excitations. As examples we consider the two-dimensional XY model and the three-dimensional compact electrodynamics.

1. Introduction

The relevance of topological excitations to the behaviour of quantum field theories and statistical mechanical systems is by now well established. Their influence on the critical behaviour of some two-dimensional systems was noticed by Kosterlitz and Thouless (1973). These authors realised that, in spite of rigorous theorems forbidding in two dimensions the spontaneous breaking of a continuous symmetry group (Mermin and Wagner 1966, Mermin 1968), the XY model does exhibit a phase transition. Actually this transition is of a different nature: it is due to the existence of vortices which behave as a Coulomb gas. This system has a dielectric low-temperature phase and a plasma high-temperature phase.

In lattice gauge theories (Wilson 1974, Kogut and Susskind 1975) there are many interesting abelian systems which possess topological excitations. Banks *et al* (1977) showed that the monopoles of three-dimensional compact quantum electrodynamics (QED) can be treated as a Coulomb gas of point-like particles, while the four-dimensional model is a gas of strings of monopoles. The introduction of matter fields transforms the Coulomb potential into a Yukawa one (Einhorn and Savit 1978).

To determine the critical behaviour of these systems it is important to take into account their topological structures explicitly. Renormalisation group (RG) calculations with this property have been carried out only for two-dimensional systems. Kosterlitz (1974) introduced a real space technique to study the Coulomb gas, and it was later applied to several systems. Elitzur *et al* (1979) used it in the Z_N spin model, while Nelson and Halperin (1980) studied liquid crystals, and Parga and Van Himbergen (1980, 1981) applied it to finite-size systems.

However, there are cases where this technique may be awkward. For a Coulomb gas of strings, such as the one appearing in four-dimensional QED, it may be very complicated to integrate over string fluctuations of a given scale.

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In this paper we present a real space RG technique with the property that the counting of fluctuations of topological excitations is done explicitly. As examples we study the two- and three-dimensional Coulomb gases, although we think it may be used in more complicated systems.

Briefly the method is as follows. We put the gas on a square (or cubic) lattice and introduced a continuous scalar field so that the potential between pairs of charges does not appear explicitly. After this, changing the scale and integrating over the inner degrees of freedom associated with the scalar field we obtain a simple expression where we can easily sum over the charge fluctuations. Since we have a lattice, this sum is trivial. For instance, a charge inside a block composed of four sites can occupy only these positions and will be seen by the others smeared over them. Clearly this effect will produce a renormalisation of the chemical potential and of the interaction.

The paper is organised as follows. In the next section we present the method, applying it to the two-dimensional Coulomb gas. The partition function is written in a convenient form to sum over charge fluctuations. We do this sum in § 3 where we also derive the RG equations. In § 4 we extend the technique to three dimensions. The last section contains our conclusions, and in an appendix we discuss a technical point.

2. The two-dimensional Coulomb gas

In this section we present our real space RG technique, applying it to the two-dimensional Coulomb gas.

There are many physical systems which can be put into this form. Among them we mention the Villain form of the two-dimensional XY model (Villain 1975), the sine-Gordon quantum field theory (Samuel 1978) and the two-dimensional superfluid (Kosterlitz and Nelson 1977).

The partition function is

$$Z = \prod_j \sum_{m(j)}{}' \exp\Big(-\mu^2 \sum_j m^2(j) + \pi K \sum_{j\neq k} m(j) \ln|j-k| m(k)\Big) \tag{2.1}$$

where $m(j)$ is the charge on the site j and can take the values $0, \pm 1$. μ^2 is the chemical potential, and K measures the coupling between pairs of charges. The prime in the sum reminds us that in two dimensions only neutral configurations contribute. The system possesses long-range forces, and their effects have to be taken into account as we iterate the RG equations.

We have placed the Coulomb gas on a square lattice, and its spacing gives the size of a charge. If we double the scale, the chemical potential has to be altered because each charge is seen by the others smeared over a block of four plaquettes of the original lattice. The coupling K has also to be changed to take into account that a configuration with a dipole inside a block renormalises a similar configuration where the dipole is absent.

Equation (2.1) can be cast into a more convenient form by introducing a scalar field defined on the lattice sites:

$$Z = \prod_j \int_{-\infty}^{\infty} \mathrm{d}\varphi(j) \sum_{m(j)} y_0^{\sum_j m^2(j)} \exp\Big(-\frac{1}{2K}\sum_{j;\nu} (\Delta_\nu \varphi)^2 + 2\pi \mathrm{i} \sum_j m(j)\varphi(j)\Big). \tag{2.2}$$

μ^2 and y_0 are related parameters, but from now on we shall use only the last one. To

obtain this expression we added new degrees of freedom but now the long-range forces are not explicit.

To start with the RG calculation we consider blocks containing four sites, as the one shown in figure 1. The x_i are the values of the scalar field on these sites.

The next step is to define the block degrees of freedom. A suitable choice is given by

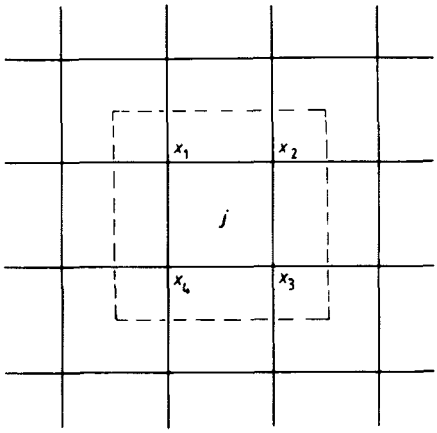

Figure 1. The four-site block for the two-dimensional lattice. The x_i are the values of the scalar field defined on those sites. j denotes the block position.

the sum of the four variables contained in the block. In this way

$$\varphi(j)=\tfrac{1}{4}\sum_{i=1}^{4} x_i \tag{2.3a}$$

$$M(j)=\sum_{i=1}^{4} m_i \tag{2.3b}$$

where j denotes the block position and the first sum was conveniently normalised. $M(j)$ is the total charge.

To simplify the algebra it is convenient to make the following change of variables:

$$\begin{aligned}\varphi_1(j)&=\tfrac{1}{2}(x_1-x_2-x_3+x_4)_j\\ \varphi_2(j)&=\tfrac{1}{2}(x_1+x_2-x_3-x_4)_j\\ \varphi_3(j)&=\tfrac{1}{2}(x_1-x_2+x_3-x_4)_j.\end{aligned} \tag{2.4}$$

The advantage of the new variables is that, together with (2.3a), they transform according to irreducible representations of the symmetry group of figure 1. φ and φ_3 are singlets, while φ_1 and φ_2 are members of a doublet. Under a rotation of 90°

$$\begin{aligned}&\varphi(j)\to\varphi(j) \qquad && \varphi_1(j)\to-\varphi_2(j)\\ &\varphi_3(j)\to-\varphi_3(j) && \varphi_2(j)\to\varphi_1(j).\end{aligned} \tag{2.5}$$

To express Z in terms of the new variables we have to evaluate two quantities. One of them is associated with the block and given by a loop

$$B(j)=(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_4)^2+(x_4-x_1)^2. \tag{2.6}$$

The other refers to the interaction between neighbouring blocks; because of cancellations, it is convenient to consider not only two blocks but the whole strips, as shown in figures 2(a) and (b). Their contribution is

$$I_x = \sum_{J_x} \{[x_1(j+\hat{x}) - x_2(j)]^2 + [x_4(j+\hat{x}) - x_3(j)]^2\}$$
$$I_y = \sum_{J_y} \{[x_4(j+\hat{y}) - x_1(j)]^2 + [x_3(j+\hat{y}) - x_2(j)]^2\}. \tag{2.7}$$

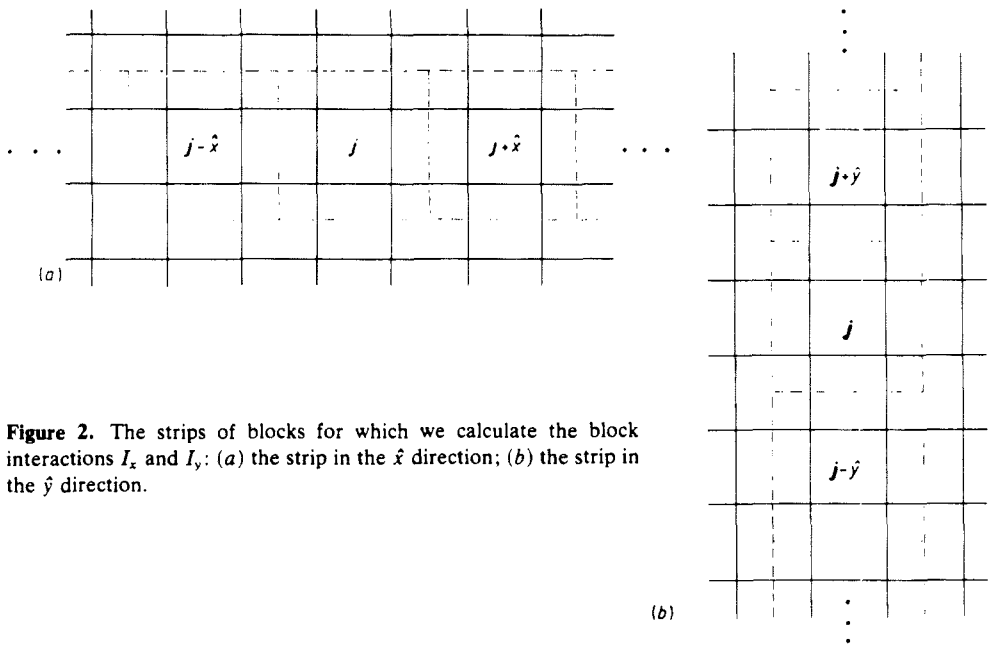

Figure 2. The strips of blocks for which we calculate the block interactions I_x and I_y: (a) the strip in the $\hat{x}$ direction; (b) the strip in the $\hat{y}$ direction.

Using (2.3a) and (2.4) the loop can be written as

$$B(j) = 2(\varphi_1^2 + \varphi_2^2)_j + 4\varphi_3^2(j) \tag{2.8}$$

and the interaction in the $\hat{x}$ direction is

$$I_x = \sum_{J_x} [\![2[\Delta_x\varphi(j)]^2 + \tfrac{1}{2}\{[\sigma_x\varphi_1(j)]^2 + [\Delta_x\varphi_2(j)]^2 + [\sigma_x\varphi_3(j)]^2\}$$
$$-2\varphi_1(j)\Delta'_x\varphi(j) + \varphi_2(j)\Delta'_x\varphi_3(j)]\!]. \tag{2.9}$$

Here we have used the notation

$$\Delta_x\varphi_i(j) = \varphi_i(j+\hat{x}) - \varphi_i(j)$$
$$\Delta'_x\varphi_i(j) = \varphi_i(j+\hat{x}) - \varphi_i(j-x) \tag{2.10}$$
$$\sigma_x\varphi_i(j) = \varphi_i(j+\hat{x}) + \varphi_i(j).$$

The interaction in the $\hat{y}$ direction can be obtained from (2.9) by a rotation of 90°. Using

the transformation properties given in (2.5) we obtain

$$I_y = \sum_{I_y} [\![2[\Delta_y\varphi(j)]^2 + \tfrac{1}{2}\{[\Delta_y\varphi_1(j)]^2 + [\sigma_y\varphi_2(j)]^2 + [\sigma_y\varphi_3(j)]^2\}$$
$$-2\varphi_2(j)\Delta'_y\varphi(j) + \varphi_1(j)\Delta'_y\varphi_3(j)]\!]. \tag{2.11}$$

From (2.8) we see that the block contributes only with diagonal terms; I_x and I_y instead have the finite differences and sums defined in (2.10). The sums, however, can be changed into differences; taking

$$\begin{aligned} &\varphi_1(j) \to (-)^{j_x}\varphi_1(j) \\ &\varphi_2(j) \to (-)^{j_y}\varphi_2(j) \\ &\varphi_3(j) \to (-)^{j_x+j_y}\varphi_3(j) \end{aligned} \tag{2.12}$$

we have that $(\sigma_x\varphi_i)^2 \to (\Delta_x\varphi_i)^2$, while the squares of the differences do not change.

Replacing $B(j)$, I_x and I_y in (2.2) we obtain

$$Z = \prod_j \int_{-\infty}^{\infty} \mathrm{d}\varphi(j) \sum_{M(j)} \exp\Big(-\frac{1}{K}\sum_{j;\nu}(\Delta_\nu\varphi)^2 + 2\pi\mathrm{i}\sum_j M(j)\varphi(j)\Big)$$
$$\times \sum_{\{m_i\}}{}' \, y_0^{\sum_l (m_1^2+m_2^2+m_3^2+m_4^2)_l} \boldsymbol{F}(\{m_i\}) \,. \tag{2.13}$$

where

$$F(\{m_i\}) = \prod_j \int_{-\infty}^{\infty} \mathrm{d}\varphi_3(j) \prod_{\alpha=1,2}\prod_j \int_{-\infty}^{\infty} \mathrm{d}\varphi_\alpha(j) \exp\Big(-\frac{1}{4K}\sum_{j;\nu}[(\Delta_\nu\varphi_\alpha)^2 + 4\varphi_\alpha^2] + \sum_j \varphi_\alpha J_\alpha\Big)$$
$$\times \exp\Big(-\frac{1}{4K}\sum_{j;\nu}[(\Delta_\nu\varphi_3)^2 + 8\varphi_3^2] + 2\pi\mathrm{i}\sum_j (-)^{j_x+j_y}\varphi_3 M_3\Big). \tag{2.14}$$

The prime in the sum over the m_i indicates that it has to be done keeping the block total charges fixed. The sources J_1 and J_2 are

$$\begin{aligned} J_1(j) &= \frac{(-)^{j_x+1}}{2K}[(-)^{j_x+j_y}\Delta'_y\varphi_3(j) - 2\Delta'_x\varphi(j) - 4\pi\mathrm{i}KM_1(j)] \\ J_2(j) &= \frac{(-)^{j_y+1}}{2K}[(-)^{j_x+j_y}\Delta'_x\varphi_3(j) - 2\Delta'_y\varphi(j) - 4\pi\mathrm{i}KM_2(j)] \end{aligned} \tag{2.15}$$

(from now on we shall use j to denote the block position).

Although the sums are over the m_i for convenience, we have introduced M_1, M_2 and M_3. These are related to the m_i through the same matrix given in (2.4).

Except for minor details, the first integral in (2.13) is already of the same form as the initial expression (2.2). To obtain the RG equations we still have to evaluate the functional F, to sum over the m_i and to absorb the result in the parameters K and y_0.

In the rest of this section we calculate F, leaving until § 3 the discussion of the other two points.

The three integrals in F are gaussian. φ_1 and φ_2 are coupled only through φ_3, and we integrate over them first. The integral over φ_1 yields the factor

$$\exp\Big(K\sum_{l,n} J_1(l)G(l-n;4)J_1(n)\Big). \tag{2.16}$$

$G(l; m^2)$ is the lattice Green function; in Fourier space it reads

$$G(\boldsymbol{k}; m^2) = \left(4 \sum_{\nu=x,y} \sin^2 k_\nu/2 + m^2\right)^{-1}. \tag{2.17}$$

There are several terms contained in the argument of (2.16). We discuss them in detail. The one quadratic in $\Delta'\varphi$ is

$$\frac{1}{K} \sum_{l,n} (-)^{l_x+n_x} \Delta'_x\varphi(l) G(l-n; 4) \Delta'_x\varphi(n), \tag{2.18}$$

and an expansion of $G(\boldsymbol{k}; m^2)$ in powers of $(k_\nu/m)^2$ gives a host of new interactions for the scalar field. However, according to the usual arguments of relevant operators (Amit 1978) we may keep only the first term; then

$$G(l-n; m^2) \simeq \delta_{l,n}/m^2 \tag{2.19}$$

and (2.18) gives a contribution

$$\exp\left(\frac{1}{4K} \sum_l [\Delta'_x\varphi(l)]^2\right) \simeq \exp\left(\frac{1}{2K} \sum_l [\Delta_x\varphi(l)]^2\right). \tag{2.20}$$

The last expression comes from an argument valid at large distances that we discuss in the appendix.

For a dilute system, charges will be several blocks apart, and the terms including $M_1(l)$ will contribute only at the same site. These are

$$-4\pi^2 K G(0; 4) \sum_l M_1^2(l) \tag{2.21}$$

and

$$-4\pi i G(0; 4) \sum_l \Delta'_x\varphi(l) M_1(l). \tag{2.22}$$

The integral over φ_2 gives similar results; the finite differences in (2.20) and (2.22) are now taken in the $\hat{y}$ direction, and M_1 has to be replaced by M_2. In both integrals there are also quadratic and linear terms in φ_3; the linear ones contribute to a source given by

$$J_3(l) = 2\pi i\left((-)^{l_x+l_y} M_3(l) + \frac{1}{2K} \Delta'_x\Delta'_y\varphi(l) + G(0; 4)[\Delta'_y M_1(l) + \Delta'_x M_2(l)]\right). \tag{2.23}$$

The last integral then yields

$$\exp\left(K \sum_{l,n} J_3(l) G^*(l-n; 8) J_3(n)\right) \tag{2.24}$$

where the Fourier transform of the Green function $G^*(l; m^2)$ is

$$G^*(\boldsymbol{k}; m^2) = \left(4 \sum_{\nu=x,y} \frac{\sin^2 k_\nu}{2} - 4G(0; 4) \sum_{\nu=x,y} \sin^2 k_\nu + m^2\right)^{-1}. \tag{2.25}$$

The difference from (2.17) comes from the term $\Delta'_\nu\varphi_3$ present in the sources J_1 and J_2. By similar considerations to those that led us to equations (2.20)–(2.22) we can express this result as

$$\exp\left(-4\pi^2 K G^*(0; 8) \sum_l \{M_3(l) + (-)^{l_x+l_y} G(0; 4)[\Delta'_y M_1(l) + \Delta'_x M_2(l)]\}^2\right). \tag{2.26}$$

For a dilute system this can be further simplified. In this case

$$M_3(l)[\Delta'_y M_1(l) + \Delta'_x M_2(l)] = 0, \tag{2.27}$$

because if there is a charge in the block l there cannot be another in a neighbouring block. In the same way we have

$$\sum_l [\Delta'_y M_1(l) + \Delta'_x M_2(l)]^2 = 2 \sum_l [M_1^2(l) + M_2^2(l)]. \tag{2.28}$$

Combining expressions (2.20)–(2.22) and (2.26)–(2.28) in (2.14) the partition function finally reads

$$Z = \prod_l \int_{-\infty}^{\infty} d\varphi(l) \sum_{M(l)} \exp\Big(-\frac{1}{2K} \sum_{l;\nu} (\Delta_\nu \varphi)^2 + 2\pi i \sum_l \varphi(l) M(l) \Big)$$
$$\times \prod_j \Big[\sum_{\{m_i\}}{}' y_0^{\sum_{i=1}^4 m_i^2} \exp\left(-2\pi^2 K [g_1(M_1^2 + M_2^2) + g_2 M_3^2]_j\right)$$
$$\times \exp\left[4\pi i G(0;4)(M_1 \Delta'_x \varphi + M_2 \Delta'_y \varphi)\right] \Big] \tag{2.29}$$

where

$$g_1 = 2G(0;4)[1 + 2G(0;4)G^*(0;8)] \qquad g_2 = 2G^*(0;8). \tag{2.30}$$

Equation (2.29) has a very simple structure. The first factor depends only on the block variables, while the second is a product over all the blocks and contains the sum over charge fluctuations with block charges fixed. Since the system is on a lattice, this sum is trivial and its result has to be absorbed in the parameters K and y_0; this we do in the next section. It should be noted that (2.29) depends only on invariants of the symmetry group of the square.

3. The renormalisation group equations

Before obtaining the RG equations we have to sum over the fluctuations of the charges inside the block. Since we have a lattice, there is a finite number of possibilities; for instance, if $M(l) = 1$ there is a single charge in the block l and it can be in any of the four sites. If we add a dipole to this block the configurations we obtain are ruled out for a dilute system. If $M(l) = 0$ up to $O(y_0^2)$ we can have either an empty block or a dipole. We do not consider objects with $M(l) = 2$ because they would have an activity $O(y_0^2)$.

Keeping only the empty block, the single charge and the dipole, the product over the blocks in (2.29) can be written

$$\prod_i \{1 + 4y_0 e^{-\nu_3}(e^{2\pi i\varphi} + e^{-2\pi i\varphi}) - 4y_0^2 (2\pi g_1)^2 (e^{-\nu_1} + e^{-\nu_2})[(\Delta_x\varphi)^2 + (\Delta_y\varphi)^2]$$
$$+ 4y_0^2 (2e^{-\nu_1} + e^{-\nu_2})\} \tag{3.1}$$

where

$$\nu_1 = 2\pi^2 K(g_1 + g_2) \qquad \nu_2 = 4\pi^2 K g_1 \qquad \nu_3 = \pi^2 K(g_1 + g_2/2). \tag{3.2}$$

To obtain this expression we have again kept only terms $(\Delta\varphi)^2$ for the scalar field and used the result of the appendix.

The term linear in y_0 corresponds to a single charge inside the block, while the quadratic one appears when there is a dipole in it. For a dilute system the last situation is very unlikely, and at most it will occur in only one block.

We can now proceed to obtain the RG equations. Consider a configuration with n charges in different blocks; its contribution to (2.29) has a factor

$$y_R^n \prod_{i=1}^{n} (e^{2\pi i\varphi(j_i)} + e^{-2\pi i\varphi(j_i)}) \tag{3.3}$$

where

$$y_R = 4e^{-\nu_3} y_0 \tag{3.4}$$

is the new activity. ν_3 appears as a result of integrating the Coulomb potential over the size of the block; it involves Green functions at the origin and contributes to the chemical potential. The factor 4 comes from the number of positions the charge can take inside the block.

The addition of a dipole in any of the blocks will give, after summing its contribution to (3.3), the constant factor

$$\exp[4y_0^2 A(2e^{-\nu_1} + e^{-\nu_2})] \tag{3.5}$$

(A is the total area) and a renormalisation to the scalar field

$$\Delta K = K_R - K = -2[8\pi G(0;4)]^2(e^{-\nu_1} + e^{-\nu_2})K^2 y_0^2 \tag{3.6}$$

which together with

$$\Delta y = y_R - y_0 = -(1 - 4e^{-\nu_3})y_0 \tag{3.7}$$

are the RG equations.

A numerical evaluation of the Green functions contained in ν_3 gives $G(0;4) = 0.13$ and $G^*(0;8) = 0.09$. From these values we calculate the zero of the right-hand side of (3.7), obtaining $K_c \simeq 0.38$. Due to the small value of $G^*(0;8)$, the largest contribution to ν_3 comes from the term $2\pi^2 KG(0;4)$.

Above K_c the line $y = 0$ consists of infrared-stable fixed points. Linearising (3.6) and (3.7) around $K = K_c$ and $y = 0$ and absorbing a constant factor in the parameters we obtain

$$\Delta y = -yx \qquad \Delta x = -y^2 \tag{3.8}$$

where x is proportional to $K - K_c$. In these expressions we recognise the finite-difference version of the differential equations found by Kosterlitz (1974).

For those values of the parameters such that y renormalises to 0, the gaussian approximation is valid and the critical index η is given by

$$\eta = \frac{1}{2\pi K_R(\infty)} \tag{3.9}$$

where $K_R(\infty)$ is K_R after an infinite number of iterations. In particular for the critical point K_c we obtain $\eta = 0.41$. This is higher than the exact value $\eta = \frac{1}{4}$; however, it seems plausible that our value can be improved by taking larger blocks. In this case $G(0; m^2)$ and $G^*(0; m^2)$ will become smaller, because their masses, which depend on the block size, will increase, giving rise to a larger K_c.

4. A three-dimensional example: compact QED

The RG technique presented in §§ 2 and 3 can be easily extended to three dimensions. To this end we consider again the Coulomb gas. An interesting system which can be written in this way is compact QED; this can be thought of as a U(1) lattice gauge theory defined by the Wilson (1974) action or as spontaneously broken continuum O(3) gauge theory. Banks *et al* (1977) in the first case and Polyakov (1977) in the second have shown that the monopoles of these systems behave as a Coulomb gas.

The phase diagram is simple, the gas is always in its plasma phase, and for compact QED this means the existence of a confining potential between electric charges (Polyakov 1977). The reason for this difference from the two-dimensional gas can be found in the different behaviour of entropy and energy of a single topological excitation: while in two dimensions both are proportional to ln R (R is the size of the system) allowing for a compensation, in the present case only the entropy has this dependence.

One starting point is the partition function written as in (2.2), but in one more dimension. The field $\varphi(j)$ and the $m(j)$ are defined on the sites of a cubic lattice. To obtain the RG equations we consider blocks containing eight sites, as the one shown in figure 3. As in two dimensions the variables x_i defined on its corners do not transform simply under the symmetry group of the cube. The orthogonal linear combinations which are the basis of irreducible representations are

$$\varphi(j) = \tfrac{1}{8} \sum_{i=1}^{8} x_i \tag{4.1}$$

$$\varphi_i(j) = A_{ij} x_j \qquad i, j = 1, 2, \ldots, 7 \tag{4.2}$$

where

$$A = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}, \tag{4.3}$$

$\varphi(j)$ is the block field and we have to integrate over the φ_i keeping it fixed. φ_1, φ_2, φ_3 and φ_4, φ_5, φ_6 are triplets, whereas φ and φ_7 are singlets.

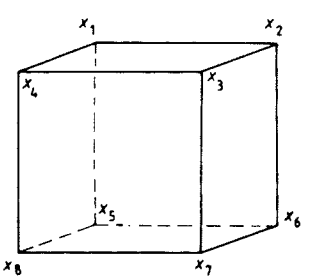

Figure 3. The eight-site block for the three-dimensional lattice. The meaning of the x_i is as in figure 1.

The quantities $B(j)$ and I_x, I_y and I_z are defined analogously to (2.6) and (2.7) and can be easily evaluated. $B(j)$ now involves all the loops shown in figure 3, but again it contains only diagonal terms in the φ_i. As to the block interactions, we only need to calculate I_x; once we know it we can find I_y and I_z using the transformation properties of the φ_i.

After a lengthy but simple algebra the partition function can be written

$$Z=\prod_l \int_{-\infty}^{\infty} d\varphi(l) \sum_{M(l)} \exp\Big(-\frac{2}{K}\sum_{l;\mu}(\Delta_\mu\varphi)^2+2\pi i\sum_l M(l)\varphi(l)\Big)\prod_l \sum_{\{m_i\}}{}' y_0^{\sum_{i=1}^8 m_i^2} F(\{m_i\}). \tag{4.4}$$

The functional F contains the integrals over the inner degrees of freedom. Those are gaussian and have mass terms which depend on the block size

$$\begin{aligned} F(\{m_i\}) = &\prod_{\alpha=4,6}\int_{-\infty}^{\infty} d\varphi_\alpha(l)\exp\Big(-\frac{1}{4K}\sum_{l;\mu}[(\Delta_\mu\varphi_\alpha)^2+8\varphi_\alpha^2]\Big)\\ &\times\exp\Big(2\pi i\sum_l[(-)^{l_x+l_y}M_4+(-)^{l_x+l_z}M_5+(-)^{l_y+l_z}M_6]_l\Big)\\ &\times\prod_{\gamma=1,3}\int_{-\infty}^{\infty} d\varphi_\gamma(l)\exp\Big(-\frac{1}{4K}\sum_{l;\mu}[(\Delta_\mu\varphi_\gamma)^2+4\varphi_\gamma^2]+\sum_l J_\gamma\varphi_\gamma\Big)\\ &\times\int_{-\infty}^{\infty} d\varphi_7(l)\exp\Big(-\frac{1}{4K}\sum_{l;\mu}[(\Delta_\mu\varphi_7)^2+12\varphi_7^2]+\sum_l J_7\varphi_7\Big). \end{aligned} \tag{4.5}$$

Here

$$\begin{aligned} J_1(l) &= -\frac{1}{2K}[(-)^{l_x}\sqrt{8}\,\Delta'_x\varphi-(-)^{l_y}\Delta'_y\varphi_4+(-)^{l_z}\Delta'_z\varphi_5-4\pi i(-)^{l_x}KM_1]\\ J_7(l) &= -\frac{1}{2K}[-(-)^{l_x}\Delta'_x\varphi_6+(-)^{l_y}\Delta'_y\varphi_5-(-)^{l_z}\Delta'_z\varphi_4-4\pi iK(-)^{l_x+l_y+l_z}M_7]. \end{aligned} \tag{4.6}$$

J_2 and J_3 can be obtained from J_1 using the transformation properties of the φ_i. $M(l)$ is the block charge

$$M(l)=\sum_{i=1}^{8} m_i \tag{4.7}$$

and the M_i are related to the m_i by the matrix (4.3).

The integrals in (4.5) are immediate and, after doing approximations similar to those used in § 2, we obtain

$$\begin{aligned} Z=&\prod_j\int_{-\infty}^{\infty} d\varphi(j)\sum_{M(j)}\exp\Big(-\frac{1}{K}\sum_{j;\mu}(\Delta_\mu\varphi)^2+2\pi i\sum_j M(j)\varphi(j)\Big)\\ &\times\prod_j\Big(\sum_{\{m_i\}} y_0^{\sum_{i=1}^8 m_i}\exp\{-2\pi^2K[h_1(M_1^2+M_2^2+M_3^2)\\ &+h_2(M_4^2+M_5^2+M_6^2)+h_3M_7^2]\}\\ &\times\exp[-2\pi i\sqrt{8}\,G(0;4)(M_1\Delta'_x\varphi+M_2\Delta'_y\varphi-M_3\Delta'_z\varphi)]\Big) \end{aligned} \tag{4.8}$$

where

$$h_1 = 2G(0;4)[1+4G(0;4)G'(0;8)] \qquad h_2 = 2G'(0;8)$$
$$h_3 = 2G(0;12)[1+6G(0;12)G'(0;8)] \tag{4.9}$$

and $G'(l;m^2)$ is the Fourier transform of

$$G'(k^2;m^2) = \left(4\sum_{\nu=x,y,z}\frac{\sin^2 k_\nu}{2} - 4G(0;4)\sum_{\nu=y,z}\sin^2 k_\nu - 4G(0;12)\sin^2 k_x + m^2\right)^{-1} \tag{4.10}$$

Except that $2K$ has been replaced by K, the first factor in (4.8) is equal to the initial expression. The second factor contains the sum over the m_i, and it distinguishes with different coefficients h_j the M_i transforming according to different irreducible representations. It is trivial to check that the last exponential has the right transformation properties. It should be noted that equations (2.29) and (4.8) have a very similar appearance.

To obtain the RG equations we proceed as in § 3, first summing over charge fluctuations inside the block and considering later the addition of a dipole as a renormalisation effect. The result is

$$K_R^{-1} = 2K^{-1}\{1+8Ky_0^2[4\pi G(0;4)]^2(e^{-\mu_1}+2e^{-\mu_2}+3e^{-\mu_3})\} \qquad y_R = 8\,e^{-\mu}y_0 \tag{4.11}$$

where

$$\mu = \pi^2 K(3h_1+3h_2+h_3) \qquad \mu_1 = \pi^2 K(h_1+2h_2+h_3)$$
$$\mu_2 = \pi^2 K(3h_1+h_3) \qquad \mu_3 = 2\pi^2 K(h_1+h_2). \tag{4.12}$$

The essential difference from the two-dimensional case is that in three (or more) dimensions there is an $O(y_0^0)$ renormalisation to K. As a consequence the fixed line at $y_0 = 0$ reduces to the point $K^{-1} = 0$.

In lattice compact QED K^{-1} is proportional to the lattice spacing and, as noticed by Banks *et al* (1977), this means that monopoles disappear in the continuum limit. As a consequence, the confining potential between electric external charges becomes a Coulomb one.

5. Conclusions

We have developed a real space RG technique able to deal with systems with topological excitations. In the cases treated, it has succeeded in reproducing all the features of the corresponding phase transitions. For the two-dimensional Coulomb gas it describes fairly well the unbinding of vortices, giving a segment of points below a critical temperature K_c^{-1} where the gaussian approximation is valid. Although the value of K_c^{-1} is too high, it could be improved by considering larger blocks. In this case the Green functions at the origin appearing in ν_3 would get higher masses and consequently their values would be appreciably reduced, giving rise to a larger K_c.

The generalisation to three dimensions was immediate and we obtained the right properties of compact QED, i.e. monopoles are dense for any non-zero value of K^{-1}. We believe that the method is also applicable to more complicated systems. Although we have restricted our analysis to Coulomb potentials, the technique could be used with

other interactions. An interesting case is the Yukawa potential, a problem which appears naturally in some lattice gauge theories (Einhorn and Savit 1978).

An interesting problem which could be treated is a gas of loops interacting via a Coulomb potential. The auxiliary field is now a vector one, and the expression equivalent to (2.2) has gauge invariance. Once the intra-block degrees of freedom associated with the vector field are integrated over, the counting of string fluctuations can be done either explicitly or in combination with a Monte Carlo simulation (Swendsen 1979).

Appendix

In this appendix we show that the approximation used in (2.20) is correct.

Strictly speaking $(\Delta'\varphi)^2$ is a new interaction generated by the RG, and we should keep it together with the initial $(\Delta\varphi)^2$ and see how it renormalises. Nevertheless this would complicate unnecessarily our calculations, and we prefer to use the following argument.

We consider the two interactions $(\Delta\varphi)^2$ and $(\Delta'\varphi)^2$ and see how they renormalise under a Migdal–Kadanoff transformation (Migdal 1975a, b, Kadanoff 1976). Figure A1 shows a block where we have to shift the inner bonds and later decimate by integrating over the y_i.

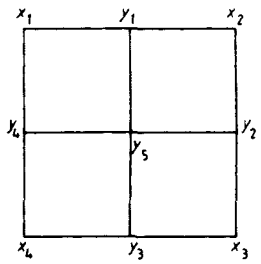

Figure A1. The block where we apply a Migdal–Kadanoff transformation. The x_i and the y_i are the values of the scalar field at the corresponding sites. We shift the bonds connecting the y_i.

After the shifting is done, the interaction $(\Delta'\varphi)^2$ in the bond (x_1, x_2) gives

$$e^{-2(x_1-x_2)^2/2K} \tag{A1}$$

while for $(\Delta\varphi)^2$ we obtain

$$\int_{-\infty}^{\infty} dy_1\, e^{-2(x_1-y_1)/2K} e^{-2(y_1-x_2)^2/2K} = e^{-(x_1-x_2)^2/2K}. \tag{A2}$$

These two results show us that after one application of the Migdal–Kadanoff transformation $(\Delta\varphi)^2$ keeps its form and its coefficient while $(\Delta'\varphi)^2$ becomes twice $(\Delta\varphi)^2$. This is the result used in (2.20).

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